



Almost additive multifractal analysis

Luis Barreira^{a,*,1}, Paulo Doutor^b

^a Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

^b Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829-516 Monte da Caparica, Portugal

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Abstract

For the class of almost additive sequences, we establish a conditional variational principle for the dimension spectra in the context of the nonadditive thermodynamic formalism. This generalizes the classical thermodynamic formalism, by replacing the topological pressure of a single function by the topological pressure of a sequence of functions. In particular, we show that each level set of the multifractal decomposition carries a full measure, that is, an ergodic invariant measure with dimension equal to the dimension of the level set. We also show that the spectra are continuous and that the irregular sets have full dimension.

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Résumé

Pour la classe des suites presque sous-additive, nous établissons un principe de variation conditionnel pour les spectres de dimension dans le cadre du formalisme thermodynamique non-additif. Ceci généralise le formalisme thermodynamique classique, en remplaçant la pression topologique d'une seule fonction par la pression topologique d'une suite des fonctions. En particulier, nous démontrons que chaque ensemble de niveau de la décomposition multifractal porte une mesure pleine, c'est-à-dire, une mesure invariante ergodique de dimension égale à la dimension de l'ensemble de niveau. Nous démontrons également que les spectres sont continus et que les ensembles irréguliers ont dimension pleine.

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1. Introduction

Our main objective is to establish a conditional variational principle for the dimension spectra of almost additive sequences. We also show that the spectra are continuous and that the irregular sets have full dimension. Our approach builds on related arguments in former work of Barreira, Saussol and Schmeling in [5], although now in the more general context of almost additive sequences.

* Corresponding author.

E-mail addresses: barreira@math.ist.utl.pt (L. Barreira), pjd@fct.unl.pt (P. Doutor).

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We recall that a sequence of functions $\Phi = (\varphi_n)_n$ is said to be *almost additive* (with respect to a transformation f) if there is a constant $C > 0$ such that for every $n, m \in \mathbb{N}$ we have:

$$-C + \varphi_n + \varphi_m \circ f^n \leq \varphi_{n+m} \leq C + \varphi_n + \varphi_m \circ f^n.$$

In particular, it was shown in [2] that for this class of sequences it is possible not only to establish a variational principle for the topological pressure, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures, among other properties. The nonadditive thermodynamic formalism developed in [1] is a generalization of the classical thermodynamic formalism (see [7,14,15]), in which the topological pressure $P(\varphi)$ of a single function φ is replaced by the topological pressure $P(\Phi)$ of a sequence of functions $\Phi = (\varphi_n)_n$. The main motivation for being introduced was to allow certain applications to a more general class of invariant sets in the dimension theory of dynamical systems. We note that the class of almost additive sequences occurs naturally in the study of multifractal spectra on nonconformal repellers (see [4]).

We formulate here a particular case of our main result (see Theorem 3). Let $f: X \rightarrow X$ be a continuous transformation of a compact metric space, and let $\Phi = (\varphi_n)_n$ be an almost additive sequence of functions $\varphi_n: X \rightarrow \mathbb{R}$. Given $\alpha \in \mathbb{R}$, we consider the level set:

$$K_\alpha = \left\{ x \in X: \lim_{n \rightarrow \infty} \frac{\varphi_n(x)}{n} = \alpha \right\}.$$

The *entropy spectrum* $\mathcal{E}: \mathbb{R} \rightarrow \mathbb{R}$ (of the sequence Φ) is defined by:

$$\mathcal{E}(\alpha) = h(f|K_\alpha),$$

where $h(f|K_\alpha)$ denotes the topological entropy of f on K_α (see Section 2.2 for the definition). We also consider the function $\mathcal{P}: \mathcal{M}_f \rightarrow \mathbb{R}$ defined by:

$$\mathcal{P}(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu,$$

in the set \mathcal{M}_f of f -invariant probability measures in X .

The following statement is a conditional variational principle for the entropy spectrum \mathcal{E} . It is an immediate consequence of Theorem 3 below.

Theorem 1. *Let f be a continuous transformation of a compact metric space X such that $\mu \mapsto h_\mu(f)$ is upper semicontinuous, and assume that the almost additive sequence Φ has a unique equilibrium measure. If $\alpha \notin \mathcal{P}(\mathcal{M}_f)$, then $K_\alpha = \emptyset$. Otherwise, if $\alpha \in \text{int } \mathcal{P}(\mathcal{M}_f)$, then $K_\alpha \neq \emptyset$, and the following properties hold:*

1. $\mathcal{E}(\alpha)$ satisfies the variational principle:

$$\mathcal{E}(\alpha) = \max \{ h_\mu(f): \mu \in \mathcal{M}_f \text{ and } \mathcal{P}(\mu) = \alpha \};$$

2. $\mathcal{E}(\alpha) = \min \{ P(q\Phi) - q\alpha: q \in \mathbb{R} \};$
3. there is an ergodic measure $\mu_\alpha \in \mathcal{M}_f$ with $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and $h_{\mu_\alpha}(f) = \mathcal{E}(\alpha)$.

In addition, the spectrum \mathcal{E} is continuous in $\text{int } \mathcal{P}(\mathcal{M}_f)$.

The theory of multifractal analysis is a subfield of the dimension theory of dynamical systems. It studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman in [9]. The first rigorous approach is due to Collet, Lebowitz and Porzio in [8] for a class of measures invariant under 1-dimensional Markov maps. In [11], Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in [13], Rand studied Gibbs measures for a class of repellers. We refer the reader to the books [3,12] for further references from the last two decades.

2. Nonadditive thermodynamic formalism

We recall in this section some notions and results from the thermodynamic formalism. We refer to [1,2,6] for details. We also establish some regularity properties of the topological pressure.

2.1. Topological pressure

We first introduce the notion of nonadditive topological pressure. Let $f : X \rightarrow X$ be a continuous transformation of a compact metric space. Given a finite open cover \mathcal{V} of X , we denote by $\mathcal{W}_n(\mathcal{V})$ the collection of vectors $V = (V_0, \dots, V_n)$ with $V_0, \dots, V_n \in \mathcal{V}$. For each $V \in \mathcal{W}_n(\mathcal{V})$, we write $m(V) = n$, and we consider the open set

$$X(V) = \bigcap_{k=0}^n f^{-k} V_k.$$

Now let Φ be a sequence of continuous functions $\varphi_n : X \rightarrow \mathbb{R}$. For each $n \in \mathbb{N}$ we define:

$$\gamma_n(\Phi, \mathcal{V}) = \sup \{ |\varphi_n(x) - \varphi_n(y)| : x, y \in X(V) \text{ for some } V \in \mathcal{W}_n(\mathcal{V}) \}.$$

We always assume that

$$\limsup_{\text{diam } \mathcal{V} \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\gamma_n(\Phi, \mathcal{V})}{n} = 0. \quad (1)$$

For each $V \in \mathcal{W}_n(\mathcal{V})$ we write:

$$\varphi(V) = \begin{cases} \sup_{X(V)} \varphi_n & \text{if } X(V) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases} \quad (2)$$

Given a set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define the function:

$$M(Z, \alpha, \Phi, \mathcal{V}) = \lim_{n \rightarrow \infty} \inf_{\Gamma} \sum_{V \in \Gamma} \exp(-\alpha m(V) + \varphi(V)),$$

where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{V})$ such that $\bigcup_{V \in \Gamma} X(V) \supset Z$. We also define

$$P_Z(\Phi, \mathcal{V}) = \inf \{ \alpha \in \mathbb{R} : M(Z, \alpha, \Phi, \mathcal{V}) = 0 \}.$$

It was shown in [1] that the limit

$$P_Z(\Phi) = \lim_{\text{diam } \mathcal{V} \rightarrow 0} P_Z(\Phi, \mathcal{V})$$

exists. The number $P_Z(\Phi)$ is called the *nonadditive topological pressure* of Φ in the set Z (with respect to f). We also write $P(\Phi) = P_X(\Phi)$. The number $h(f|Z) = P_Z(0)$ is called the *topological entropy* of f in Z .

Now let \mathcal{M}_f be the set of all f -invariant probability measures in X . We recall that a measure μ in X is said to be *f-invariant* if $\mu(f^{-1}A) = \mu(A)$ for every measurable set $A \subset X$. Given $\mu \in \mathcal{M}_f$ and a measurable partition ξ of X , we define:

$$H_\mu(\xi) = - \sum_{C \in \xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. The *Kolmogorov–Sinai entropy* of f with respect to μ is given by:

$$h_\mu(f) = \sup \{ h_\mu(f, \xi) : H_\mu(\xi) < \infty \},$$

where

$$h_\mu(f, \xi) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\xi_n),$$

with ξ_n the partition of X into sets $C_1 \cap C_2 \cap \dots \cap C_n$, where $C_k \in f^{-k+1}\xi$ for $k = 1, \dots, n$. We note that for each measure $\mu \in \mathcal{M}_f$, the limit

$$\liminf_{\text{diam } \mathcal{V} \rightarrow 0} \{ h(Z, \mathcal{V}) : \mu(Z) = 1 \} \quad (3)$$

exists, where

$$h(Z, \mathcal{V}) = \inf \{ \alpha \in \mathbb{R} : M(Z, \alpha, 0, \mathcal{V}) = 0 \}.$$

Proposition 1. (See [12].) *Let f be a continuous transformation of a compact metric space. If f preserves an ergodic probability measure μ , then the limit in (3) is equal to $h_\mu(f)$.*

2.2. u -dimension

We recall here a notion introduced by Barreira and Schmeling in [6]. Let again $f : X \rightarrow X$ be a continuous transformation of a compact metric space, and let \mathcal{V} be a finite open cover of X . Let also $u : X \rightarrow \mathbb{R}$ be a positive continuous function. Given a set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define the function

$$N(Z, \alpha, u, \mathcal{V}) = \lim_{n \rightarrow \infty} \inf_{\Gamma} \sum_{V \in \Gamma} \exp(-\alpha u(V)),$$

where $u(V)$ is defined as in (2), and where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{V})$ such that $\bigcup_{u \in \Gamma} X(V) \supset Z$. We also define:

$$\dim_{u, \mathcal{V}} Z = \inf \{ \alpha \in \mathbb{R} : N(Z, \alpha, u, \mathcal{V}) = 0 \}.$$

We can show that the limit

$$\dim_u Z = \lim_{\text{diam } \mathcal{V} \rightarrow 0} \dim_{u, \mathcal{V}} Z$$

exists. The number $\dim_u Z$ is called the u -dimension of the set Z (with respect to f). We note that if $u = 1$, then $\dim_u Z$ is equal to the topological entropy $h(f|Z)$ of f in Z .

The following result is an easy consequence of the definitions.

Proposition 2. *The number $\dim_u Z = \alpha$ is the unique root α of the equation $P_Z(-\alpha U) = 0$, where $U = (u_n)_n$ with $u_n = \sum_{k=0}^{n-1} u \circ f^k$ for each $n \in \mathbb{N}$.*

Furthermore, given a probability measure μ in X , we set:

$$\dim_{u, \mathcal{V}} \mu = \inf \{ \dim_{u, \mathcal{V}} Z : \mu(Z) = 1 \}.$$

We can show that the limit

$$\dim_u \mu = \lim_{\text{diam } \mathcal{V} \rightarrow 0} \dim_{u, \mathcal{V}} \mu$$

exists, and we call it the u -dimension of μ . Moreover, the *lower* and *upper u -pointwise dimensions* of μ at the point $x \in X$ are defined by:

$$\underline{d}_{\mu, u}(x) = \lim_{\text{diam } \mathcal{V} \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_V - \frac{\log \mu(X(V))}{u(V)},$$

and

$$\bar{d}_{\mu, u}(x) = \lim_{\text{diam } \mathcal{V} \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_V - \frac{\log \mu(X(V))}{u(V)},$$

where the infimum and supremum are taken over all vectors $V \in \mathcal{W}_n(\mathcal{V})$ such that $x \in X(V)$. If $\mu \in \mathcal{M}_f$ is ergodic, then (see [6])

$$\dim_u \mu = \underline{d}_{\mu, u}(x) = \bar{d}_{\mu, u}(x) = \frac{h_\mu(f)}{\int_X u d\mu},$$

for μ -almost every $x \in X$.

3. Almost additive sequences

We say that a sequence Φ of functions $\varphi_n : X \rightarrow \mathbb{R}$ is *almost additive* (with respect to f) if there is a constant $C > 0$ such that for every $m, n \in \mathbb{N}$ and $x \in X$ we have:

$$-C + \varphi_n(x) + \varphi_m(f^n(x)) \leq \varphi_{m+n}(x) \leq C + \varphi_n(x) + \varphi_m(f^n(x)).$$

The almost additive sequences occur naturally for example in the study of nonconformal repellers (see [4]). The following are some basic properties.

Proposition 3. (See [2, Theorem 3].) *If $f : X \rightarrow X$ is a continuous transformation of a compact metric space, $\Phi = (\varphi_n)_n$ is an almost additive sequence, and $\mu \in \mathcal{M}_f$, then the limit*

$$\varphi(x) = \lim_{n \rightarrow \infty} \frac{\varphi_n(x)}{n}$$

exists for μ -almost every $x \in X$. Furthermore:

1. $\frac{\varphi_n}{n} \rightarrow \varphi$ in $L^1(X, \mu)$ when $n \rightarrow \infty$;
2. $\frac{1}{n} \int_X \varphi_n d\mu \rightarrow \int_X \varphi d\mu$ when $n \rightarrow \infty$;
3. the function,

$$\mathcal{M}_f \ni \mu \mapsto \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu,$$

is continuous with the weak topology in \mathcal{M}_f .*

In addition, we can easily show that the function φ is f -invariant (and not only μ -almost everywhere f -invariant). We also recall the variational principle established in [2, Theorem 13].

Proposition 4. *If f is a continuous transformation of a compact metric space X , and Φ is an almost additive sequence of continuous functions in X , then*

$$\begin{aligned} P(\Phi) &= \sup \left\{ h_\mu(f) + \int_X \lim_{n \rightarrow \infty} \frac{\varphi_n(x)}{n} d\mu(x) : \mu \in \mathcal{M}_f \right\} \\ &= \sup \left\{ h_\mu(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu : \mu \in \mathcal{M}_f \right\}. \end{aligned}$$

A measure $\mu \in \mathcal{M}_f$ is called an *equilibrium measure* (with respect to f) of the almost additive sequence Φ if,

$$P(\Phi) = h_\mu(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu.$$

We denote by $A(X)$ the family of almost additive sequences of continuous functions (satisfying (1)). It is shown in [2] that if the function $\mu \mapsto h_\mu(f)$ is upper semicontinuous, then every sequence in $A(X)$ has an equilibrium measure.

We proceed by establishing some regularity properties of the topological pressure. Let $E(X) \subset A(X)$ be the family of sequences with a unique equilibrium measure.

Theorem 2. *If $f : X \rightarrow X$ is a continuous transformation of a compact metric space and $\mu \mapsto h_\mu(f)$ is upper semicontinuous, then the following properties hold:*

1. given $\Phi \in A(X)$, the function $t \mapsto P(\Phi + t\Psi)$ is differentiable at $t = 0$ for every $\Psi \in A(X)$ if and only if $\Phi \in E(X)$; in this case the unique equilibrium measure μ_Φ of Φ is ergodic, and

$$\frac{d}{dt}P(\Phi + t\Psi)|_{t=0} = \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_\Phi; \quad (4)$$

2. for each open set $A \subset \mathbb{R}$, if $\Phi + t\Psi \in E(X)$ for every $t \in A$, then the function $t \mapsto P(\Phi + t\Psi)$ is of class C^1 in A .

Proof. We follow partially arguments in [10]. Let $t \in \mathbb{R}$ and $\Phi, \Psi \in A(X)$. Then $\Phi + t\Psi \in A(X)$ and this sequence has equilibrium measures. Let $\mu_t, \mu_\Phi \in \mathcal{M}_f$ be equilibrium measures respectively of $\Phi + t\Psi$ and Φ . By Proposition 4 we have:

$$\begin{aligned} P(\Phi + t\Psi) - P(\Phi) &\geq h_{\mu_\Phi}(f) + \lim_{n \rightarrow \infty} \int_X \frac{\varphi_n + t\psi_n}{n} d\mu_\Phi - P(\Phi) \\ &= t \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_\Phi, \end{aligned}$$

and

$$\begin{aligned} P(\Phi + t\Psi) - P(\Phi) &= P(\Phi + t\Psi) - P((\Phi + t\Psi) - t\Psi) \\ &\leq P(\Phi + t\Psi) - h_{\mu_t}(f) - \lim_{n \rightarrow \infty} \int_X \frac{\varphi_n + t\psi_n - t\psi_n}{n} d\mu_t \\ &= t \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_t. \end{aligned}$$

This yields the inequalities

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_\Phi &\leq \frac{P(\Phi + t\Psi) - P(\Phi)}{t} \leq \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_t, \\ \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_\Phi &\geq \frac{P(\Phi + t\Psi) - P(\Phi)}{t} \geq \lim_{n \rightarrow \infty} \int_X \frac{\psi_n}{n} d\mu_t, \end{aligned} \quad (5)$$

respectively when $t > 0$ and $t < 0$.

Now we assume that the function $t \mapsto P(\Phi + t\Psi)$ is differentiable at $t = 0$. Let μ_Φ and ν_Φ be two equilibrium measures of Φ . Given a continuous function $a : X \rightarrow \mathbb{R}$, we consider the sequence $A = (a_n)_n$ with

$$a_n = \sum_{k=0}^{n-1} a \circ f^k,$$

for each $n \in \mathbb{N}$. By Birkhoff's ergodic theorem, we have:

$$\int_X a d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X a_n d\mu,$$

for every $\mu \in \mathcal{M}_f$. Therefore, since $t \mapsto P(\Phi + tA)$ is differentiable at $t = 0$, it follows from (5) that

$$\begin{aligned} \int_X a d\mu_\Phi &= \lim_{n \rightarrow \infty} \int_X \frac{a_n}{n} d\mu_\Phi = \lim_{t \rightarrow 0} \frac{P(\Phi + tA) - P(\Phi)}{t} \\ &= \lim_{n \rightarrow \infty} \int_X \frac{a_n}{n} d\nu_\Phi = \int_X a d\nu_\Phi. \end{aligned}$$

Since the function a is arbitrary, we conclude that $\mu_\Phi = \nu_\Phi$.

For the converse we start with an auxiliary statement.

Lemma 1. *The following properties hold:*

1. if $\mu_{t_n} \rightarrow \mu$ when $n \rightarrow \infty$, for some sequence $t_n \rightarrow 0$, then μ is an equilibrium measure of Φ ;
2. if $\Phi \in E(X)$, then μ is the unique equilibrium measure μ_Φ of Φ .

Proof. We consider a measure μ as in statement 1. By Proposition 4 we have:

$$P(\Phi) \geq h_\mu(f) + \lim_{m \rightarrow \infty} \frac{1}{m} \int_X \varphi_m d\mu. \quad (6)$$

On the other hand, by Proposition 3 the function

$$\mu \mapsto \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu$$

is continuous. Since $\mu \mapsto h_\mu(f)$ is upper semicontinuous, the function

$$\mu \mapsto h_\mu(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu$$

is also upper semicontinuous, and thus,

$$\begin{aligned} h_\mu(f) + \lim_{m \rightarrow \infty} \frac{1}{m} \int_X \varphi_m d\mu &\geq \limsup_{n \rightarrow \infty} \left(h_{\mu_{t_n}}(f) + \lim_{m \rightarrow \infty} \frac{1}{m} \int_X \varphi_m d\mu_{t_n} \right) \\ &= \limsup_{n \rightarrow \infty} \left(P(\Phi + t_n \Psi) - t_n \lim_{m \rightarrow \infty} \frac{1}{m} \int_X \psi_m d\mu_{t_n} \right). \end{aligned} \quad (7)$$

Moreover,

$$\begin{aligned} P(\Phi + t_n \Psi) &= \sup \left\{ h_\mu(f) + \lim_{m \rightarrow \infty} \frac{1}{m} \int_X (\varphi_m + t_n \psi_m) d\mu : \mu \in \mathcal{M}_f \right\} \\ &\geq \sup \left\{ h_\mu(f) + \lim_{m \rightarrow \infty} \frac{1}{m} \int_X \varphi_m d\mu : \mu \in \mathcal{M}_f \right\} \\ &\quad - \sup \left\{ - \lim_{m \rightarrow \infty} \frac{1}{m} \int_X t_n \psi_m d\mu : \mu \in \mathcal{M}_f \right\} \\ &= P(\Phi) - \sup \left\{ - \lim_{m \rightarrow \infty} \frac{1}{m} \int_X t_n \psi_m d\mu : \mu \in \mathcal{M}_f \right\}. \end{aligned} \quad (8)$$

Since the sequence $(\psi_n)_n$ is almost additive, we have:

$$-C(m-1) + \sum_{k=0}^{m-1} \psi_1(f^k(x)) \leq \psi_m(x) \leq \sum_{k=0}^{m-1} \psi_1(f^k(x)) + C(m-1),$$

and thus

$$\|\psi_m\|_\infty \leq m(\|\psi_1\|_\infty + C), \quad (9)$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Therefore, there exists $A > 0$ such that for every $\mu \in \mathcal{M}_f$ and $m \in \mathbb{N}$ we have:

$$\left| \int_X \frac{\psi_m}{m} d\mu \right| \leq \frac{\|\psi_m\|_\infty}{m} \leq A.$$

We conclude that

$$\sup \left\{ - \lim_{m \rightarrow \infty} \frac{1}{m} \int_X t_n \psi_m d\mu : \mu \in \mathcal{M}_f \right\} \leq |t_n| A.$$

On other hand, for every $t \in \mathbb{R}$ we have:

$$-t \lim_{m \rightarrow \infty} \int_X \frac{\psi_m}{m} d\mu \geq -|t| A,$$

and from (7) and (8) it follows that

$$h_\mu(f) + \lim_{m \rightarrow \infty} \int_X \frac{\varphi_m}{m} d\mu \geq P(\Phi) - 2|t_n| A.$$

Since $t_n \rightarrow 0$ we obtain:

$$h_\mu(f) + \lim_{m \rightarrow \infty} \int_X \frac{\varphi_m}{m} d\mu \geq P(\Phi),$$

which together with (6) yields the identity

$$h_\mu(f) + \lim_{m \rightarrow \infty} \int_X \frac{\psi_m}{m} d\mu = P(\Phi).$$

This establishes the first statement.

The second statement follows immediately from the first one. \square

Now take $\Phi \in E(X)$, $\Psi \in A(X)$, and $t \in \mathbb{R}$. Let μ_Φ be the unique equilibrium measure of Φ , and let μ_t be an equilibrium measure of $\Phi + t\Psi$. By Lemma 1, we have $\mu_t \rightarrow \mu_\Phi$ when $t \rightarrow 0$, and hence it follows from (5) that

$$\lim_{t \rightarrow 0} \frac{P(\Phi + t\Psi) - P(\Phi)}{t} = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n d\mu_\Phi.$$

This establishes the first statement in the theorem, including identity (4).

Now we establish the second statement. Assume that Φ has a unique equilibrium measure μ_Φ . We show that μ_Φ is ergodic. Otherwise it would exist an f -invariant measurable set $Y \subset X$ with $0 < \mu_\Phi(Y) < 1$. Consider the f -invariant probability measures ν_1 and ν_2 defined by:

$$\nu_1(B) = \frac{\mu_\Phi(B \cap Y)}{\mu_\Phi(Y)} \quad \text{and} \quad \nu_2(B) = \frac{\mu_\Phi(B \cap (X \setminus Y))}{\mu_\Phi(X \setminus Y)},$$

for every measurable $B \subset X$. By Proposition 3 we have:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_Y \varphi_n d\mu_\Phi = \mu_\Phi(Y) \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\nu_1,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{X \setminus Y} \varphi_n d\mu_\Phi = \mu_\Phi(X \setminus Y) \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\nu_2.$$

Since μ_Φ is an equilibrium measure of Φ , we obtain:

$$\begin{aligned}
P(\Phi) &= h_{\mu_\Phi}(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\mu_\Phi \\
&= \mu_\Phi(Y) h_{\nu_1}(f) + \mu_\Phi(X \setminus Y) h_{\nu_2}(f) \\
&\quad + \mu_\Phi(Y) \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\nu_1 + \mu_\Phi(X \setminus Y) \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\nu_2 \\
&\leq \max_{i=1,2} \left\{ h_{\nu_i}(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n d\nu_i \right\} \leq P(\Phi).
\end{aligned}$$

Therefore, at least one of the measures ν_1 or ν_2 is an equilibrium measure of Φ , and thus it must be equal to μ_Φ . But $\nu_i \neq \mu_\Phi$ for $i = 1, 2$. This contradiction shows that μ_Φ is ergodic.

Now take an open set $A \subset \mathbb{R}$ such that $\Phi + t\Psi \in E(X)$ for every $t \in A$. For each $s \in A$ we have:

$$\begin{aligned}
\frac{d}{dt} P(\Phi + t\Psi)|_{t=s} &= \frac{d}{dt} P(\Phi + s\Psi + (t-s)\Psi)|_{t=s=0} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n d\mu_{\Phi+s\Psi},
\end{aligned}$$

and $t \mapsto P(\Phi + t\Psi)$ is differentiable in A . The continuity of the derivative follows from statement 4 in Proposition 3 together with Lemma 1. \square

4. Almost additive multifractal analysis

This section contains our results concerning the u -dimension spectrum of almost additive sequences. In particular, we establish a conditional variational principle and we show that the irregular sets have full u -dimension.

4.1. Conditional variational principle

We formulate our main result in this section: a conditional variational principle for the u -dimension spectrum. This contains as a particular case Theorem 1 in the introduction for the entropy spectrum.

Let $d \in \mathbb{N}$ and take $(A, B) \in A(X)^d \times A(X)^d$. We write

$$A = (\Phi^1, \dots, \Phi^d) \quad \text{and} \quad B = (\Psi^1, \dots, \Psi^d),$$

and also $\Phi^i = (\varphi_n^i)_n$ and $\Psi^i = (\psi_n^i)_n$. We assume that

$$\liminf_{m \rightarrow \infty} \frac{\psi_m^i(x)}{m} > 0 \quad \text{and} \quad \psi_n^i(x) > 0 \quad (10)$$

for every $i = 1, \dots, d$, $x \in X$, and $n \in \mathbb{N}$. Given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ we define:

$$K_\alpha = \bigcap_{i=1}^d \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{\varphi_n^i(x)}{\psi_n^i(x)} = \alpha_i \right\}. \quad (11)$$

The function $\mathcal{F}_u : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha,$$

is called the u -dimension spectrum of the pair (A, B) (with respect to f). We also consider the function $\mathcal{P} : \mathcal{M}_f \rightarrow \mathbb{R}^d$ defined by:

$$\begin{aligned}
\mathcal{P}(\mu) &= \left(\frac{\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n^1 d\mu}{\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^1 d\mu}, \dots, \frac{\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n^d d\mu}{\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^d d\mu} \right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{\int_X \varphi_n^1 d\mu}{\int_X \psi_n^1 d\mu}, \dots, \frac{\int_X \varphi_n^d d\mu}{\int_X \psi_n^d d\mu} \right).
\end{aligned} \quad (12)$$

Proposition 3 ensures that the second identity in (12) holds, and that the function \mathcal{P} is continuous. Since \mathcal{M}_f is compact and connected, the set $\mathcal{P}(\mathcal{M}_f)$ is also compact and connected.

The following is a conditional variational principle for the spectrum \mathcal{F}_u . Given vectors $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ in \mathbb{R}^d we use the notations:

$$\alpha * \beta = (\alpha_1 \beta_1, \dots, \alpha_d \beta_d) \quad \text{and} \quad \langle \alpha, \beta \rangle = \sum_{i=1}^d \alpha_i \beta_i.$$

We also consider the sequence of functions $U = (u_n)_n$ with $u_n = \sum_{k=0}^{n-1} u \circ f^k$ for each $n \in \mathbb{N}$.

Theorem 3 (Conditional variational principle). *Let f be a continuous transformation of a compact metric space X such that $\mu \mapsto h_\mu(f)$ is upper semicontinuous, and assume that*

$$\text{span}\{\Phi^1, \Psi^1, \dots, \Phi^d, \Psi^d, U\} \subset E(X).$$

If $\alpha \notin \mathcal{P}(\mathcal{M}_f)$, then $K_\alpha = \emptyset$. Otherwise, if $\alpha \in \text{int } \mathcal{P}(\mathcal{M}_f)$, then $K_\alpha \neq \emptyset$, and the following properties hold:

1. $\mathcal{F}_u(\alpha)$ satisfies the variational principle:

$$\mathcal{F}_u(\alpha) = \max \left\{ \frac{h_\mu(f)}{\int_X u d\mu} : \mu \in \mathcal{M}_f \text{ and } \mathcal{P}(\mu) = \alpha \right\}; \quad (13)$$

2. $\mathcal{F}_u(\alpha) = \min\{T_u(\alpha, q) : q \in \mathbb{R}^d\}$, where $T_u(\alpha, q)$ is the unique real number satisfying:

$$P(\langle q, A - \alpha * B \rangle - T_u(\alpha, q)U) = 0; \quad (14)$$

3. there is an ergodic measure $\mu_\alpha \in \mathcal{M}_f$ with $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u d\mu_\alpha} = \mathcal{F}_u(\alpha). \quad (15)$$

In addition, the spectrum \mathcal{F}_u is continuous in $\text{int } \mathcal{P}(\mathcal{M}_f)$.

The proof of Theorem 3 is given in Section 4.3.

We note that $T_u(\alpha, q)$ is defined implicitly by (14). By Theorem 2, the function

$$(p, \alpha, q) \mapsto P(\langle q, A - \alpha * B \rangle - pU)$$

is of class C^1 . By the Implicit function theorem, we conclude that $(\alpha, q) \mapsto T_u(\alpha, q)$ is also of class C^1 in $\mathbb{R}^d \times \mathbb{R}^d$, since by (4)

$$\frac{\partial}{\partial p} P(\langle q, A - \alpha * B \rangle - pU) \Big|_{(p,q)=(T_u(\alpha,q),q)} = - \int_X u d\mu_q < 0,$$

where μ_q is the unique equilibrium measure of $\langle q, A - \alpha * B \rangle - T_u(\alpha, q)U$.

4.2. Irregular sets

This section is dedicated to the study of irregular sets, that is, sets for which the limits in (11) do not exist. We consider the particular case of topological Markov chains.

Given $p \in \mathbb{N}$, we define the shift map $\sigma : X \rightarrow X$ in $X = \{1, \dots, p\}^{\mathbb{N}}$ by $\sigma(i_1 i_2 \dots) = (i_2 i_3 \dots)$. Given $\beta > 1$, we introduce a distance in X by:

$$d((i_1 i_2 \dots), (j_1 j_2 \dots)) = \sum_{k=1}^{\infty} \beta^{-k} |i_k - j_k|.$$

The space X is compact with respect to this distance. Given a $p \times p$ matrix A with entries a_{ij} equal to 0 or 1, we consider the compact subset $\Sigma_A \subset X$ formed by the sequences $(i_1 i_2 \dots) \in X$ such that $a_{i_n i_{n+1}} = 1$ for every $n \in \mathbb{N}$. The restriction $\sigma|_{\Sigma_A}$ is called a *topological Markov chain*. We say that $\sigma|_{\Sigma_A}$ is *topologically mixing* if $A^k > 0$ for some $k \in \mathbb{N}$.

We consider the *irregular set*:

$$I = \bigcap_{i=1}^d \left\{ x \in \Sigma_A : \liminf_{n \rightarrow \infty} \frac{\varphi_n^i(x)}{\psi_n^i(x)} < \limsup_{n \rightarrow \infty} \frac{\varphi_n^i(x)}{\psi_n^i(x)} \right\},$$

and we denote by m_u the equilibrium measure of u , when it is unique.

Theorem 4. *Let $\sigma|_{\Sigma_A}$ be a topologically mixing topological Markov chain. If*

$$\text{span}\{\Phi^1, \Psi^1, \dots, \Phi^d, \Psi^d, U\} \subset E(\Sigma_A),$$

and $\mathcal{P}(m_u) \in \text{int } \mathcal{P}(\mathcal{M}_\sigma)$, then

$$\dim_u I = \dim_u \Sigma_A.$$

Proof. Take $\varepsilon > 0$ and $\alpha^0 \in \text{int } \mathcal{P}(\mathcal{M}_\sigma)$. For $i = 1, \dots, d$, let $\alpha^i \in \text{int } \mathcal{P}(\mathcal{M}_\sigma)$ be such that $\alpha_i^0 \neq \alpha_i^i$ and

$$\mathcal{F}_u(\alpha^i) > \mathcal{F}_u(\alpha^0) - \varepsilon. \quad (16)$$

This is always possible in view of the continuity of the spectrum \mathcal{F}_u (see Theorem 3). Moreover, by statement 3 in Theorem 3, for each $i = 0, \dots, d$ there is an ergodic measure $\mu^i \in \mathcal{M}_\sigma$ with $\mathcal{P}(\mu^i) = \alpha^i$ such that

$$\mu^i(K_{\alpha^i}) = 1 \quad \text{and} \quad \dim_u \mu^i = \mathcal{F}_u(\alpha^i).$$

Then the measures $\mu^0, \mu^1, \dots, \mu^d$ form a distinguishing collection of measures (in the sense of [6]) for the sequences $(\varphi_n^i/\psi_n^i)_n, i = 1, \dots, d$. It follows from Theorem 7.2 in [6] that

$$\dim_u I \geq \min\{\dim_u \mu^0, \dim_u \mu^1, \dots, \dim_u \mu^d\},$$

which together with (16) yields that

$$\dim_u I \geq \mathcal{F}_u(\alpha^0) - \varepsilon.$$

The arbitrariness of ε and α^0 implies that

$$\dim_u I \geq \sup_{\alpha \in \text{int } \mathcal{P}(\mathcal{M}_\sigma)} \mathcal{F}_u(\alpha). \quad (17)$$

By Proposition 2, we have $P(-(\dim_u \Sigma_A)U) = 0$. Therefore,

$$0 = P(-(\dim_u \Sigma_A)U) = h_{m_u}(\sigma) - \dim_u \Sigma_A \int_{\Sigma_A} u \, dm_u,$$

and thus,

$$\dim_u m_u = \frac{h_{m_u}(\sigma)}{\int_{\Sigma_A} u \, dm_u} = \dim_u \Sigma_A.$$

Moreover, since $m_u(K_{\mathcal{P}(m_u)}) = 1$ we have $\dim_u m_u \leq \dim_u K_{\mathcal{P}(m_u)}$, and

$$\mathcal{F}_u(\mathcal{P}(m_u)) \geq \dim_u m_u = \dim_u \Sigma_A.$$

Since $\mathcal{P}(m_u) \in \text{int } \mathcal{P}(\mathcal{M}_\sigma)$, together with (17) this yields the statement in the theorem. \square

4.3. Proof of Theorem 3

We first establish several auxiliary results. The assumptions in Theorem 3 will be standing assumptions in this section.

Lemma 2. *If $\alpha \in \mathcal{P}(\mathcal{M}_f)$, then*

$$\inf_{q \in \mathbb{R}^d} P(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U) \geq 0.$$

Proof. Given $\delta > 0$ and $\tau \in \mathbb{N}$ we consider the sets:

$$L_{\delta, \tau} = \{x \in X: \|A_n(x) - \alpha * B_n(x)\| < \delta n \text{ for every } n \geq \tau\},$$

where

$$A_n = (\varphi_n^1, \dots, \varphi_n^d) \quad \text{and} \quad B_n = (\psi_n^1, \dots, \psi_n^d),$$

with the norm $\|\alpha\| = |\alpha_1| + \dots + |\alpha_d|$ in \mathbb{R}^d . For each $x \in K_\alpha$ and $i = 1, \dots, d$ we have:

$$\lim_{n \rightarrow \infty} \frac{\varphi_n^i(x)}{\psi_n^i(x)} = \alpha_i.$$

Since $\psi_n^i > 0$, for each $\delta > 0$ there exists $\tau \in \mathbb{N}$ (independent of i) such that for every $n > \tau$ we have:

$$\left| \frac{\varphi_n^i(x)}{\psi_n^i(x)} - \alpha_i \right| < \frac{\delta}{d \max_i (\|\psi_1^i\|_\infty + C)}.$$

Using (9) we obtain:

$$\|A_n(x) - \alpha * B_n(x)\| = \sum_{i=1}^d |\varphi_n^i(x) - \alpha_i \psi_n^i(x)| < \frac{\delta}{d \max_i (\|\psi_1^i\|_\infty + C)} \sum_{i=1}^d \psi_n^i(x) < \delta n.$$

This shows that $x \in L_{\delta, \tau}$, and hence,

$$K_\alpha \subset \bigcap_{\delta > 0} \bigcup_{\tau \in \mathbb{N}} L_{\delta, \tau}.$$

On the other hand, by (1), given $\delta > 0$ we have:

$$\limsup_{n \rightarrow \infty} \frac{\gamma_n(\Phi^i, \mathcal{V})}{n} < \frac{\delta}{d},$$

for any finite open cover \mathcal{V} of X with sufficiently small diameter. Therefore, $\gamma_n(\Phi^i, \mathcal{V})/n < \delta/d$ for any sufficiently large n . This shows that

$$|\varphi_n^i(x) - \varphi_n^i(y)| < \delta n/d$$

for every $V \in \mathcal{W}_n(\mathcal{V})$ and $x, y \in X(V)$. Now let $A(V) = (\Phi^1(V), \dots, \Phi^d(V))$, where $\Phi^i(V)$ is defined as in (2). For each $y \in X(V)$ we have:

$$\|A(V) - A_n(y)\| = \sum_{i=1}^d |\Phi^i(V) - \varphi_n^i(y)| \leq \sum_{i=1}^d \sup_{x \in X(V)} |\varphi_n^i(x) - \varphi_n^i(y)| \leq \delta n,$$

and analogously,

$$\|B(V) - B_n(y)\| \leq \delta n,$$

for every $V \in \mathcal{W}_n(\mathcal{V})$ and $y \in X(V)$.

Given $q \in \mathbb{R}^d$, $V \in \mathcal{W}_n(\mathcal{V})$ such that $X(V) \cap L_{\delta, \tau} \neq \emptyset$, and $y \in X(V) \cap L_{\delta, \tau}$, we obtain:

$$\begin{aligned}
-\langle q, A - \alpha * B \rangle(V) &\leq |\langle q, A(V) - \alpha * B(V) \rangle| \\
&\leq \|q\| \cdot \|A(V) - \alpha * B(V)\| \\
&\leq \|q\| \cdot \|A(V) - A_n(y)\| + \|q\| \cdot \|\alpha * B_n(y) - \alpha * B(V)\| \\
&\quad + \|q\| \cdot \|A_n(y) - \alpha * B_n(y)\| \\
&\leq \|q\|(\delta n + \|\alpha\|\delta n + \delta n) = (2 + \|\alpha\|)\|q\|\delta n.
\end{aligned}$$

Therefore, setting $T = -\mathcal{F}_u(\alpha)U$ we have:

$$\begin{aligned}
\exp(T(V) - \beta n) &= \exp([T + \langle q, A - \alpha * B \rangle](V) - \langle q, A - \alpha * B \rangle(V) - \beta n) \\
&\leq \exp([\langle q, A - \alpha * B \rangle + T](V) - [\beta - (2 + \|\alpha\|)\|q\|\delta]n),
\end{aligned}$$

for every $\beta \in \mathbb{R}$. Now take $k \geq \tau$ and $\Gamma \subset \bigcup_{n \geq k} \mathcal{W}_n(\mathcal{V})$ such that $L_{\delta, \tau} \subset \bigcup_{V \in \Gamma} X(V)$. Without loss of generality we assume that there is no $V \in \Gamma$ with $X(V) \cap L_{\delta, \tau} = \emptyset$. We obtain:

$$\begin{aligned}
&\sum_{V \in \Gamma} \exp(T(V) - \beta m(V)) \\
&\leq \sum_{V \in \Gamma} \exp([\langle q, A - \alpha * B \rangle + T](V) - [\beta - (2 + \|\alpha\|)\|q\|\delta]m(V)),
\end{aligned}$$

and hence,

$$M(L_{\delta, \tau}, \beta, T, \mathcal{V}) \leq M(L_{\delta, \tau}, \beta - (2 + \|\alpha\|)\|q\|\delta, \langle q, A - \alpha * B \rangle + T, \mathcal{V}).$$

Since

$$\begin{aligned}
&\inf\{\beta: M(L_{\delta, \tau}, \beta - (2 + \|\alpha\|)\|q\|\delta, \langle q, A - \alpha * B \rangle + T, \mathcal{V}) = 0\} \\
&= \inf\{\gamma: M(L_{\delta, \tau}, \gamma, \langle q, A - \alpha * B \rangle + T, \mathcal{V}) = 0\} + (2 + \|\alpha\|)\|q\|\delta,
\end{aligned}$$

letting $\text{diam } \mathcal{V} \rightarrow 0$ we obtain

$$P_{L_{\delta, \tau}}(T) \leq P_{L_{\delta, \tau}}(\langle q, A - \alpha * B \rangle + T) + (2 + \|\alpha\|)\delta\|q\|$$

for every $\delta > 0$ and $q \in \mathbb{R}^d$. By Proposition 2 we have $P_{K_\alpha}(T) = 0$. Moreover, it was proved in [1] that if $Z_1 \subset Z_2 \subset X$, then $P_{Z_1}(\Phi) \leq P_{Z_2}(\Phi)$, and that for a countable family of sets $\{Z_i: i \in \mathbb{N}\}$ we have:

$$P_{\bigcup_{i \in \mathbb{N}} Z_i}(\Phi) = \sup_{i \in \mathbb{N}} P_{Z_i}(\Phi).$$

Therefore,

$$\begin{aligned}
0 = P_{K_\alpha}(T) &\leq P_{\bigcup_{\tau \in \mathbb{N}} L_{\delta, \tau}}(T) = \sup_{\tau \in \mathbb{N}} P_{L_{\delta, \tau}}(T) \\
&\leq \sup_{\tau \in \mathbb{N}} P_{L_{\delta, \tau}}(\langle q, A - \alpha * B \rangle + T) + (2 + \|\alpha\|)\delta\|q\| \\
&\leq P(\langle q, A - \alpha * B \rangle + T) + (2 + \|\alpha\|)\delta\|q\|,
\end{aligned}$$

for every $\delta > 0$ and $q \in \mathbb{R}^d$. The statement in the lemma follows from the arbitrariness of δ . \square

Lemma 3. If $\alpha \in \text{int } \mathcal{P}(\mathcal{M}_f)$, then

$$\min_{q \in \mathbb{R}^d} P(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U) = 0,$$

and there exists an ergodic equilibrium measure $\mu_\alpha \in \mathcal{M}_f$ with $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u d\mu_\alpha} = \mathcal{F}_u(\alpha).$$

Proof. Let $r > 0$ be the distance of α to $\mathbb{R}^d \setminus \mathcal{P}(\mathcal{M}_f)$. Take $q \in \mathbb{R}^d$ and define:

$$F(q) = P(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U).$$

Given $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ with $\beta_i = \alpha_i + r \operatorname{sgn} q_i / (2d)$ for each i , we have:

$$\|\beta - \alpha\| = \sum_{i=1}^d |\beta_i - \alpha_i| = \sum_{i=1}^d \left| \frac{1}{2d} r \operatorname{sgn} q_i \right| = \frac{r}{2} < r,$$

and hence $\beta \in \mathcal{P}(\mathcal{M}_f)$. Therefore, there exists $\mu \in \mathcal{M}_f$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_X A_n d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \beta * B_n d\mu,$$

and we obtain:

$$\begin{aligned} F(q) &\geq h_\mu(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X [\langle q, A_n - \alpha * B_n \rangle - \mathcal{F}_u(\alpha)u_n] d\mu \\ &= h_\mu(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \left\langle q, \int_X (A_n - \alpha * B_n) d\mu \right\rangle - \mathcal{F}_u(\alpha) \int_X u d\mu. \end{aligned} \quad (18)$$

Since

$$\begin{aligned} \left\langle q, \int_X (\beta - \alpha) * B_n d\mu \right\rangle &= \sum_{i=1}^d q_i \int_X (\beta_i - \alpha_i) \psi_n^i d\mu = \sum_{i=1}^d \frac{1}{2d} r q_i \operatorname{sgn} q_i \int_X \psi_n^i d\mu \\ &= \frac{1}{2d} r \sum_{i=1}^d |q_i| \int_X \psi_n^i d\mu, \end{aligned}$$

we have:

$$\frac{1}{n} \left\langle q, \int_X (A_n - \alpha * B_n) d\mu \right\rangle = \frac{1}{2d} r \sum_{i=1}^d |q_i| \frac{1}{n} \int_X \psi_n^i d\mu + \frac{1}{n} \left\langle q, \int_X (A_n - \beta * B_n) d\mu \right\rangle.$$

Taking the limit when $n \rightarrow \infty$, we obtain:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left\langle q, \int_X (A_n - \alpha * B_n) d\mu \right\rangle &= \frac{1}{2d} r \sum_{i=1}^d |q_i| \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^i d\mu \\ &\geq \frac{1}{2d} r \|q\| \min_i \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^i d\mu. \end{aligned} \quad (19)$$

Since $h_\mu(f) \geq 0$, from (18) and (19) it follows that

$$F(q) \geq \frac{1}{2d} r \|q\| \min_i \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^i d\mu - \mathcal{F}_u(\alpha) \int_X u d\mu. \quad (20)$$

We note that the right-hand side of inequality (20) takes arbitrarily large values for $\|q\|$ sufficiently large. Thus, there exists $R \in \mathbb{R}$ such that $F(q) \geq F(0)$ for every $q \in \mathbb{R}^d$ with $\|q\| \geq R$.

Since F is of class C^1 (by Theorem 2), it attains a minimum at some point $q = q(\alpha)$ with $\|q(\alpha)\| \leq R$, thus satisfying $DF(q(\alpha)) = 0$. Let μ_α be the equilibrium measure of the sequence of functions

$$\langle q(\alpha), A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U.$$

By Theorem 2 we have:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_X (A_n - \alpha * B_n) d\mu_\alpha = DF(q(\alpha)) = 0, \quad (21)$$

and thus $\mathcal{P}(\mu_\alpha) = \alpha$. Moreover,

$$F(q(\alpha)) = h_{\mu_\alpha}(f) - \mathcal{F}_u(\alpha) \int_X u d\mu_\alpha.$$

By Lemma 2, we have $F(q(\alpha)) \geq 0$ and hence,

$$\mathcal{F}_u(\alpha) \leq \frac{h_{\mu_\alpha}(f)}{\int_X u d\mu_\alpha}.$$

Again by Theorem 2, the measure μ_α is ergodic, and thus

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u d\mu_\alpha} \geq \mathcal{F}_u(\alpha).$$

On other hand, by (21) we have:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_X A_n d\mu_\alpha = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \alpha * B_n d\mu_\alpha,$$

and from Proposition 3 it follows that $\mu_\alpha(K_\alpha) = 1$. Hence,

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha \geq \dim_u \mu_\alpha,$$

and we conclude that $\dim_u \mu_\alpha = \mathcal{F}_u(\alpha)$. Therefore,

$$\begin{aligned} \min_{q \in \mathbb{R}^d} P_\Phi(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U) &= F(q(\alpha)) = h_{\mu_\alpha}(\Phi) - \mathcal{F}_u(\alpha) \int_X u d\mu_\alpha \\ &= h_{\mu_\alpha}(\Phi) - \frac{h_{\mu_\alpha}(\Phi)}{\int_X u d\mu_\alpha} \int_X u d\mu_\alpha = 0. \end{aligned}$$

This completes the proof of the lemma. \square

Proof of Theorem 3. Take $\alpha \in \mathbb{R}^d$ with $K_\alpha \neq \emptyset$ and let $x \in K_\alpha$. Then

$$\lim_{n \rightarrow \infty} \frac{\varphi_n^i(x)}{\psi_n^i(x)} = \alpha_i \quad \text{for } i = 1, \dots, d. \quad (22)$$

We consider the sequence $(\mu_n)_n$ of probability measures in X defined by:

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}.$$

The set $V(x)$ of all sublimits of this sequence in the weak* topology is nonempty, and clearly $V(x) \subset \mathcal{M}_f$.

Take $\delta > 0$, an ergodic measure $\mu \in V(x)$ (taking an ergodic decomposition, we can always assume that μ is ergodic), and $i \in \{1, \dots, d\}$. Proceeding as in the proof of Theorem 3 in [2] we obtain a sequence $(n_{k,i})_k$ of natural numbers such that

$$-\delta + \frac{\varphi_{n_{k,i}}^i(x)}{n_{k,i}} \leq \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n^i d\mu \leq \delta + \frac{\varphi_{n_{k,i}}^i(x)}{n_{k,i}},$$

and

$$-\delta + \frac{\psi_{n_{k,i}}^i(x)}{n_{k,i}} \leq \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^i d\mu \leq \delta + \frac{\psi_{n_{k,i}}^i(x)}{n_{k,i}}.$$

By (10), there exist $q \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$-\delta + \frac{\psi_m(x)}{m} > 0$$

for every $m \geq q$ and $\delta \in (0, \delta_0)$. Since δ can be made arbitrarily small, it follows from (22) that

$$\frac{\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \varphi_n^i d\mu}{\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \psi_n^i d\mu} = \alpha_i.$$

Therefore,

$$\mathcal{P}(\mu) = \lim_{n \rightarrow \infty} \left(\frac{\int_X \varphi_n^1 d\mu}{\int_X \psi_n^1 d\mu}, \dots, \frac{\int_X \varphi_n^d d\mu}{\int_X \psi_n^d d\mu} \right) = \alpha,$$

and $\alpha \in \mathcal{P}(\mathcal{M}_f)$.

Now take $\alpha \in \text{int } \mathcal{P}(\mathcal{M}_f)$ and $\mu \in \mathcal{M}_f$ with $\mathcal{P}(\mu) = \alpha$. By the upper semicontinuity of the function,

$$\mu \mapsto \frac{h_\mu(f)}{\int_X u d\mu},$$

the existence of the maximum in (13) follows readily from the compactness of \mathcal{M}_f together with the continuity of \mathcal{P} . To show that the maximum coincides with $\mathcal{F}_u(\alpha)$, we notice that by Lemma 3 and Proposition 4,

$$\begin{aligned} 0 &= \inf_{q \in \mathbb{R}^d} P(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U) \\ &\geq \inf_{q \in \mathbb{R}^d} \left\{ h_\mu(f) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_X [\langle q, A_n - \alpha * B_n \rangle - \mathcal{F}_u(\alpha)u_n] d\mu \right\}. \end{aligned}$$

Since $\mathcal{P}(\mu) = \alpha$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \langle q, A_n - \alpha * B_n \rangle d\mu = 0$$

and thus,

$$0 \geq \inf_{q \in \mathbb{R}^d} \left\{ h_\mu(f) - \mathcal{F}_u(\alpha) \int_X u d\mu \right\} = h_\mu(f) - \mathcal{F}_u(\alpha) \int_X u d\mu.$$

Therefore,

$$\frac{h_\mu(f)}{\int_X u d\mu} \leq \mathcal{F}_u(\alpha).$$

On other hand, by Lemma 3 there exists an ergodic equilibrium measure μ_α with $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u d\mu_\alpha} = \mathcal{F}_u(\alpha).$$

This establishes identities (13) and (15) (in particular $K_\alpha \neq \emptyset$).

Now take $q(\alpha) \in \mathbb{R}^d$ such that

$$P(\langle q(\alpha), A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U) = 0.$$

Proceeding as in the proof of Lemma 3 we obtain

$$\mathcal{F}_u(\alpha) = T_u(\alpha, q(\alpha)) \geq \inf \{ T_u(\alpha, q) : q \in \mathbb{R}^d \}.$$

On other hand, again by Lemma 3 and the definition of T_u we have:

$$P(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha)U) \geq 0 = P(\langle q, A - \alpha * B \rangle - T_u(\alpha, q)U)$$

for every $q \in \mathbb{R}^d$. Therefore,

$$\mathcal{F}_u(\alpha) \leq \inf\{T_u(\alpha, q) : q \in \mathbb{R}^d\}.$$

This establishes property 2.

Now we establish the continuity of the spectrum. Take $\alpha \in \text{int } \mathcal{P}(\mathcal{M}_f)$ and let $(\alpha_n)_n$ be a sequence in $\text{int } \mathcal{P}(\mathcal{M}_f)$ converging to α . For each $n \in \mathbb{N}$, take $q_n \in \mathbb{R}^d$ such that

$$\mathcal{F}_u(\alpha_n) = T_u(\alpha_n, q_n).$$

We also consider $q(\alpha) \in \mathbb{R}^d$ such that

$$\mathcal{F}_u(\alpha) = T_u(\alpha, q(\alpha)).$$

The existence of these vectors is guaranteed by statement 2 in Theorem 3. The same statement implies that

$$T_u(\alpha_n, q_n) = \min_{q \in \mathbb{R}^d} T_u(\alpha_n, q) \leq T_u(\alpha_n, q(\alpha)),$$

and hence,

$$\limsup_{n \rightarrow \infty} \mathcal{F}_u(\alpha_n) \leq \mathcal{F}_u(\alpha).$$

On other hand, since $\mathcal{F}_u(\alpha)$ is a minimum, it follows from the continuity of the function T_u (which in fact is of class C^1) that

$$\liminf_{n \rightarrow \infty} \mathcal{F}_u(\alpha_n) = \liminf_{n \rightarrow \infty} T_u(\alpha_n, q_n) \geq \mathcal{F}_u(\alpha).$$

This shows that $\mathcal{F}_u(\alpha_n) \rightarrow \mathcal{F}_u(\alpha)$ when $n \rightarrow \infty$, and the proof of the theorem is complete. \square

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